Core 4 June 2008 Answers

1) a) $\frac{(2x^2-7x-4)(x+1)}{(3x^2+x-2)(x-4)} = \frac{(2x+1)\cancel{(x-4)}\cancel{(x+1)}}{(3x-2)\cancel{(x+1)}\cancel{(x-4)}} = \frac{2x+1}{3x-2}$

b)

$$\begin{array}{r} x-2 \\ x^2+4x+1\overline{\big)\,x^3+2x^2-6x-5} \\ \underline{x^3+4x^2+x} \\ -2x^2-7x-5 \\ \underline{-2x^2-8x-2} \\ x-3 \end{array}$$

$\text{Quotient} = x-2$

$\text{Remainder} = x-3$

2) u = ln(x) $\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = x^4$ so v = $\frac{1}{5}x^5$ Use by parts (product) rule for integration

$$\int_1^e x^4 \ln x\,dx = \left[\ln(x)\frac{1}{5}x^5\right]_1^e - \int_1^e \frac{1}{5}x^5 \times \frac{1}{x}dx = \frac{1}{5}e^5 - \int_1^e \frac{1}{5}x^4 dx = \frac{1}{5}e^5 - \left[\frac{1}{25}x^5\right]_1^e$$

$$= \frac{1}{5}e^5 - \frac{1}{25}e^5 + \frac{1}{25} = \frac{4}{25}e^5 + \frac{1}{25} = \frac{1}{25}(4e^5+1)$$

3) i)

$$x^2y - xy^2 = 2$$
$$2xydx + x^2dy - y^2dx - 2xydy = 0$$
$$(x^2-2xy)dy = (y^2-2xy)dx$$
$$\frac{dy}{dx} = \frac{y^2-2xy}{x^2-2xy}$$

ii) a) $\frac{dy}{dx} = 0$ $\therefore y^2 - 2xy = 0$

$y(y-2x) = 0$

$y = 0 \quad y = 2x$

b)

$$x^2 \times 2x - x(2x)^2 = 2$$
$$-2x^3 = 2$$
$$x^3 = -1$$
$$x = -1 \quad y = -2$$

4) i)

$$AB = \begin{pmatrix}1\\3\\4\end{pmatrix} - \begin{pmatrix}3\\2\\3\end{pmatrix} = \begin{pmatrix}-2\\1\\1\end{pmatrix} \qquad r = \begin{pmatrix}3\\2\\3\end{pmatrix} + s\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

ii) Let OP is perpendicular to AB therefore $OP \bullet AB = 0$

$$OP = \begin{pmatrix}3-2s\\2+s\\3+s\end{pmatrix}$$

$$(3-2s)\times(-2) + (2+s)\times 1 + (3+s)\times 1 = 0$$
$$-6 + 4s + 2 + s + 3 + s = 0$$
$$6s - 1 = 0 \qquad s = \frac{1}{6}$$

Point P $2\frac{2}{3}i + 2\frac{1}{6}j + 3\frac{1}{6}k \quad = \begin{pmatrix}2\frac{2}{3}\\2\frac{1}{6}\\3\frac{1}{6}\end{pmatrix}$

5) i) $\sqrt{\dfrac{1-x}{1+x}} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$

$$(1-x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(-x)+\frac{\left(\frac{1}{2}\right)\times\left(-\frac{1}{2}\right)}{2!}(-x)^2\ldots = 1-\frac{1}{2}x-\frac{1}{8}x^2\ldots$$

$$(1+x)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(x)+\frac{\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)}{2!}(x)^2\ldots. = 1-\frac{1}{2}x+\frac{3}{8}x^2\ldots$$

$$\left(1-\frac{1}{2}x-\frac{1}{8}x^2\right)\left(1-\frac{1}{2}x+\frac{3}{8}x^2\right) = 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2\ldots = 1-x+\frac{1}{2}x^2$$

ii) $x=\dfrac{2}{7}$ $\quad \sqrt{\dfrac{1-\cancel{2}/\cancel{7}}{1+\cancel{2}/\cancel{7}}} = \sqrt{\dfrac{5/7}{9/7}} = \sqrt{\dfrac{5}{9}} = \dfrac{1}{3}\sqrt{5}$

$$\frac{1}{3}\sqrt{5} = 1-\frac{2}{7}+\frac{1}{2}\left(\frac{2}{7}\right)^2 = \frac{37}{49} \qquad \therefore \sqrt{5} = 3\times\frac{37}{49} = \frac{111}{49}$$

6) i)

$1+2t = 12+s$ (1)

$3t = -4s$ (2)

$-5+4t = 5-2s$ (3)

$(2)\Rightarrow t = -\dfrac{4}{3}s$ $\qquad 1+2\left(-\dfrac{4}{3}s\right) = 12+s$

$s=-3$

$t=4$

Check in (3) $\quad -5+4\times 4 = 5-2\times -3$

$11 = 11$

works in (3) therefore intersects

x = 9 y = 12 z = 11

ii)

$$\begin{pmatrix}2\\3\\4\end{pmatrix}\bullet\begin{pmatrix}1\\-4\\-2\end{pmatrix} = 2\times 1+3\times -4+4\times -2 = 2-12-8 = -18$$

$|a| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$ $\qquad |b| = \sqrt{1^2+4^2+2^2} = \sqrt{21}$

$\cos\theta = \dfrac{-18}{\sqrt{29}\sqrt{21}}$ $\qquad \theta = 137°$ $\qquad$ (acute = 43°)

7) i) $y = \operatorname{cosec}x = \dfrac{1}{\sin x}$

$$\frac{dy}{dx} = \frac{\sin x\times 0-1\times\cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x}\times\frac{\cos x}{\sin x} = -\operatorname{cosec}x\cot x$$

ii) $\dfrac{dx}{dt} = -\sin x\tan x\cot x$

$$\int\frac{1}{-\sin x\tan x}dx = \int\cot t\,dt$$

$$\int -\operatorname{cosec}x\tan x\,dx = \int\cot t\,dt$$

$$\operatorname{cosec}x = \ln|\sin t|+c$$

$x = \frac{\pi}{6}$ $\quad t = \frac{\pi}{2}$ $\quad \operatorname{cosec}\left(\frac{\pi}{6}\right) = \ln\left|\sin\left(\frac{\pi}{2}\right)\right| + c$

$c = 2$ $\quad \operatorname{cosec} x = \ln|\sin t| + 2$

8) i) $\frac{2t}{(t+1)^2} = \frac{A}{t+1} + \frac{B}{(t+1)^2}$ $\quad t = -1 \quad -2 = B$ $\quad \frac{2}{t+1} - \frac{2}{(t+1)^2}$

$2t = A(t+1) + B$ $\quad t = 0 \quad 0 = A - 2$

$A = 2$

ii) $t = (2x-1)^{\frac{1}{2}}$

$\frac{dt}{dx} = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \times 2 = (2x-1)^{-\frac{1}{2}}$ $\quad x = \frac{1}{2}(t^2+1)$

$dx = (2x-1)^{\frac{1}{2}} dt$

$$\int \frac{1}{x+\sqrt{2x-1}}dx = \int \frac{1}{x+t}(2x-1)^{\frac{1}{2}}dt = \int \frac{1}{\frac{1}{2}(t^2+1)+t} t\,dt = \int \frac{2t}{t^2+2t+1}dt = \int \frac{2t}{(t+1)^2}dt$$

iii) $\int_1^5 \frac{1}{x+\sqrt{2x-1}}dx = \int_1^3 \frac{2t}{(t+1)^2}dt = \int_1^3 \frac{2}{t+1} - \frac{2}{(t+1)^2}dt = \left[2\ln(t+1) + \frac{2}{t+1}\right]_1^3$

x = 5 t = 3
x = 1 t = 1

$$= \left(2\ln 4 + \frac{1}{2}\right) - (2\ln 2 + 1) = 4\ln 2 + \frac{1}{2} - 2\ln 2 - 1 = 2\ln 2 - \frac{1}{2}$$

9) i) $y = 4\sin\theta$ max = 4 when $\sin\theta = 1 \; \therefore \theta = \frac{\pi}{2}$ (point A)

$y = 4\sin\theta = 0 \quad \therefore \sin\theta = 0 \quad \theta = 0, \pi, 2\pi \quad \therefore \theta = 2\pi$ (point B)

ii) $\frac{dx}{d\theta} = 2 + 2\cos 2\theta$ $\quad \frac{dy}{d\theta} = 4\cos\theta$

$$\frac{dy}{dx} = \frac{4\cos\theta}{2+2\cos 2\theta} = \frac{4\cos\theta}{2+2(2\cos^2\theta - 1)} = \frac{4\cos\theta}{4\cos^2\theta} = \frac{1}{\cos\theta} = \sec\theta$$

iii) $\frac{dy}{dx} = 2$ $\quad \sec\theta = 2$ $\quad \cos\theta = \frac{1}{2}$ $\quad \theta = \frac{\pi}{3}$

$\therefore$ point C $\quad \theta = -\frac{\pi}{3}$

$x = 2\left(-\frac{\pi}{3}\right) + \sin\left(2 \times -\frac{\pi}{3}\right)$ $\quad y = 4\sin\left(-\frac{\pi}{3}\right)$

$x = -\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ $\quad y = -2\sqrt{3}$